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Abstract
Jacobi algebroids (i.e. 'Jacobi versions' of Lie algebroids) are studied in the context of graded Jacobi brackets on graded commutative algebras. This unifies various concepts of graded Lie structures in geometry and physics. A method of describing such structures by classical Lie algebroids via certain gauging (in the spirit of E Witten's gauging of the exterior derivative) is developed. One constructs a corresponding Cartan differential calculus (graded commutative one) in a natural manner. This, in turn, gives canonical generating operators for triangular Jacobi algebroids. One gets, in particular, the Lichnerowicz–Jacobi homology operators associated with classical Jacobi structures. Courant–Jacobi brackets are obtained in a similar way and used to define an abstract notion of a Courant–Jacobi algebroid and Dirac–Jacobi structure.

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1. Introduction

In this paper, we propose a unification of various concepts of graded brackets one meets in geometry and physics and a method of 'gauging' which allows us to pass from the world of derivations (i.e. tangent bundles, vector fields, exterior derivatives, Lie algebroids) to the world of first-order differential operators in the spirit of Witten's [Wi] gauging of the exterior derivative. This algebra and geometry is noncommutative in the sense that bosonic and fermionic parts are incorporated in a unique scheme. We concentrate on purely mathematical aspects to keep the size of the paper readable, but we hope that possible applications to the

Batalin–Vilkovisky formalism, BRST-method, integrability and Dirac structures, etc, will be found.

For a vector bundle E over the base manifold M, let $A(E) = \oplus_{k \in \mathbb{Z}} A^k(E)$ be the exterior algebra of multisections of E. This is a basic geometric model for a graded associative commutative algebra with unity. We will refer to elements of $\Omega^k(E) = A^k(E^*)$ as to k-forms on E. Here, we identify $A^0(E) = \Omega^0(E)$ with the algebra $C^\infty(M)$ of smooth functions on the base and $A^k(E) = \{0\}$ for $k < 0$. Denote by $|X|$ the Grassmann degree of the multisection $X \in A(E)$.

As has been observed in [KS], a *Lie algebroid* structure on E (for the traditional definition and main properties we recommend the survey article [Ma]) can be identified with a *Gerstenhaber algebra* structure (in the terminology of [KS]) on $A(E)$ which is just a *graded Poisson bracket* on $A(E)$ of degree -1. Recall that a graded Poisson bracket of degree k on a $\mathbb{Z}$-graded associative commutative algebra $\mathcal{A} = \oplus_{i \in \mathbb{Z}} \mathcal{A}^i$ is a graded bilinear map

$$\{\cdot, \cdot\} : \mathcal{A} \times \mathcal{A} \to \mathcal{A} \tag{1}$$

of degree k (i.e. $|\{a, b\}| = |a| + |b| + k$) such that

1. $\{a, b\} = -(-1)^{(a+k)(b+k)}\{b, a\}$ (graded anticommutativity),
2. $\{a, bc\} = \{a, b\}c + (-1)^{(a+k)b}b\{a, c\}$ (graded Leibniz rule),
3. $\{\{a, b\}, c\} = \{a, \{b, c\}\} - (-1)^{(a+k)(b+k)}\{b, \{a, c\}\}$ (graded Jacobi identity).

Here we use the convention that we write just a for $|a|$.

It is obvious that this notion extends naturally to more general gradings in the algebra. The Leibniz rule tells that the Poisson bracket is identified with a skew biderivation Λ on $\mathcal{A}$ ('bivector field'), $\{a, b\} = \Lambda(a, b)$, which, due to the Jacobi identity, has a special property ('the Schouten bracket $[\![\Lambda, \Lambda]\!]$ vanishes'). A precise meaning for the notions in the quotation marks has been given in [Kr2]. With respect to the standard terminology, the graded derivation $\Lambda_a = \{a, \cdot\}$ associated with $a \in \mathcal{A}$ is called the corresponding *Hamiltonian vector field* and the map $a \mapsto \Lambda_a$ is a homomorphism from $(\mathcal{A}, \{\cdot, \cdot\})$ into the graded Lie algebra $(\mathrm{Der}(\mathcal{A}), [\![\cdot, \cdot]\!])$ of graded derivations of $\mathcal{A}$ with the graded commutator $[\![\cdot, \cdot]\!]$.

For a graded commutative algebra with unity $\mathbf{1}$, a natural generalization of a graded Poisson bracket is *graded Jacobi bracket*. The only difference is that we replace the Leibniz rule by

$$\{a, bc\} = \{a, b\}c + (-1)^{(a+k)b}b\{a, c\} - \{a, \mathbf{1}\}bc \tag{2}$$

which just means that $\{a, \cdot\}$ is a first-order differential operator on $\mathcal{A}$ (for the differential calculus on graded commutative algebras we refer to [ViA, VK]). This goes back to the well-known observation by Kirillov [Ki] that in the case of $\mathcal{A} = C^\infty(M)$ every local Lie bracket is of first order (an algebraic version of this fact in the ungraded case has been proved in [Gr]). A graded associative commutative algebra with a graded Jacobi structure we will call a *graded Jacobi algebra*.

Definition. *A graded Jacobi algebra is a graded, say $\mathbb{Z}^n$-graded, associative commutative algebra $\mathcal{A} = \oplus_{i \in \mathbb{Z}^n}$ with unit $\mathbf{1}$ equipped with a graded Jacobi bracket $\{\cdot, \cdot\}$ of degree $k \in \mathbb{Z}^n$, i.e. a graded bilinear map*

$$\{\cdot, \cdot\} : \mathcal{A} \times \mathcal{A} \to \mathcal{A} \tag{3}$$

of degree k (i.e. $|\{a, b\}| = |a| + |b| + k$) such that

1. *$\{a, b\} = -(-1)^{\langle a+k, b+k \rangle}\{b, a\}$ (graded anticommutativity),*
2. *$\{a, bc\} = \{a, b\}c + (-1)^{\langle a+k, b \rangle}b\{a, c\} - \{a, \mathbf{1}\}bc$ (graded generalized Leibniz rule),*
3. *$\{\{a, b\}, c\} = \{a, \{b, c\}\} - (-1)^{\langle a+k, b+k \rangle}\{b, \{a, c\}\}$ (graded Jacobi identity),*

where $\langle \cdot, \cdot \rangle$ is the standard pairing in $\mathbb{Z}^n$.

The generalized Leibniz rule tells that the bracket is a bidifferential operator on $\mathcal{A}$ of first order. In the non-graded case, under the assumption that there are no non-trivial nilpotent elements in $\mathcal{A}$, every Lie bracket given by a bidifferential operator is known [Gr] to be of first order (it is a generalization of this result for $\mathcal{A} = C^\infty(M)$ [Ki]). In the classical case of the algebra $C^\infty(M)$, every skew-symmetric first-order bidifferential operator J splits into $J = \Lambda + I \wedge \Gamma$, where Λ is a bivector field, Γ is a vector field and I is identity, so that the corresponding bracket of functions reads

$$\{f, g\}_J = \Lambda(f, g) + f\Gamma(g) - g\Gamma(f). \tag{4}$$

The Jacobi identity for this bracket is usually written in terms of the Schouten–Nijenhuis bracket by

$$[\![\Gamma, \Lambda]\!] = 0 \tag{5}$$

$$[\![\Lambda, \Lambda]\!] = -2\Gamma \wedge \Lambda. \tag{6}$$

Hence, every Jacobi bracket on $C^\infty(M)$ can be identified with the pair $J = (\Lambda, \Gamma)$ satisfying the above conditions, i.e. with a *Jacobi structure* on M (cf [Li]). Note that we use the version of the Schouten–Nijenhuis bracket which gives a graded Lie algebra structure on multivector fields (cf [Mi]) and which differs from the classical one by signs. On the other hand, this Jacobi identity can be written in terms of the algebraic Richardson–Nijenhuis bracket (cf [NR]) $[\![J, J]\!]^{\mathrm{RN}} = 0$ of skew-multiplicative maps on $C^\infty(M)$ which, as has been observed in [GM], is a deformation of a Schouten-type bracket, when reduced to first-order *polydifferential operators* on $C^\infty(M)$, i.e. skew-symmetric multidifferential operators. However, this bracket on first-order polydifferential operators is not the Schouten bracket for the Lie algebroid of linear scalar first-order differential operators but a Jacobi version of it. This means that here we have a difference similar to the difference between Poisson and Jacobi brackets in the classical case, if we understand the Schouten bracket as being a graded Poisson bracket according to [Kr2].

An analogous construction for a general Lie algebroid has been introduced in [IM1] under the name of a *generalized Lie algebroid*. We have recognized this structure as being an odd Jacobi structure in [GM] (in fact, a graded Jacobi bracket of degree -1 in the terminology of this paper), where the name *Jacobi algebroid* has been used. These structures are closely related to the concept of Lie-like brackets on affine bundles [GGU, MMS], as has been mentioned in [GGU].

The generalized Lie algebroids in the terminology of [IM1] or the Jacobi algebroids in the terminology of [GM] are associated with the pairs combined with a Lie algebroid bracket on a vector bundle E over M and a 1-cocycle $\Phi \in \Omega^1(E)$, $\mathrm{d}\Phi = 0$, relative to the Lie algebroid exterior derivative d. The Schouten–Jacobi bracket on the graded algebra $A(E)$ of multisections of E is given by

$$[\![X, Y]\!]^\Phi = [\![X, Y]\!] + xX \wedge i_\Phi Y - (-1)^x y i_\Phi X \wedge Y \tag{7}$$

where we use the convention that $x = |X| - 1$ is the shifted degree of X in the graded algebra $A(E)$ and $[\![\cdot, \cdot]\!]$ is the Schouten bracket of the corresponding Lie algebroid. Note that $i_\Phi X = (-1)^x [\![X, \mathbf{1}]\!]^\Phi$ and (2) is satisfied:

$$[\![X, Y \wedge Z]\!]^\Phi = [\![X, Y]\!]^\Phi \wedge Z + (-1)^{x(y+1)} Y \wedge [\![X, Z]\!]^\Phi - [\![X, \mathbf{1}]\!]^\Phi \wedge Y \wedge Z. \tag{8}$$

The Jacobi algebroid bracket (7) can be written in the form

$$\begin{aligned}
[\![X, Y]\!]^\Phi &= [\![X, Y]\!] + |X|X \wedge i_\Phi Y - (-1)^x |Y| i_\Phi X \wedge Y + (-1)^x i_\Phi(X \wedge Y) \\
&= [\![X, Y]\!] + (\mathrm{Deg} \wedge i_\Phi)(X, Y) - X \wedge i_\Phi Y + (-1)^x i_\Phi X \wedge Y
\end{aligned} \tag{9}$$

where $\mathrm{Deg}(X) = |X|X$ is a derivation of $A(E)$ and $\mathrm{Deg} \wedge i_\Phi$ is an appropriate wedge product of graded derivations. This shows that the above bracket is essentially like (4) with $\Lambda = [\![\cdot, \cdot]\!] + \mathrm{Deg} \wedge i_\Phi$ and $\Gamma = -i_\Phi$.

Note that the bracket $[\![\cdot, \cdot]\!]^\Phi$ is completely determined by its values on the Lie subalgebra $C^\infty(M) \oplus \mathrm{Sec}(E)$ due to the generalized Leibniz property (8). The Lie algebra $C^\infty(M) \oplus \mathrm{Sec}(E)$ has the grading induced from $A(E)$, but we must stress that it can be viewed as a standard Lie algebra with a grading and not a graded Lie algebra (the antisymmetry and Jacobi identity is standard, not graded). We show in the next section that there is a *linear Jacobi structure* on E^* which corresponds to the Lie algebra structure $(C^\infty(M) \oplus \mathrm{Sec}(E), [\![\cdot, \cdot]\!]^\Phi)$. This is a Jacobi analogue of the correspondence

$$\text{Lie algebroid structure on } E \quad \leftrightarrow \quad \text{linear Poisson structure on } E^*. \qquad (10)$$

Since *linearity* of tensors on vector bundles is a particular case of *homogeneity* in the sense of [DLM], we will study also relations between homogeneous Poisson and Jacobi structures.

In section 3 we develop a method of inducing Jacobi algebroid brackets by gauging. This is an analogue of Witten's gauging of the exterior derivative

$$\mathrm{e}^{-f} \, \mathrm{d}(\mathrm{e}^f \mu) = \mathrm{d}\mu + \mathrm{d}f \wedge \mu \qquad (11)$$

for the Schouten–Nijenhuis bracket (the role of the differential (11) in studying Jacobi structures has already been observed by Lichnerowicz [Li]). We get in this way an appropriate concept of a Lie differential which gives generating operators of BV-algebras associated with generalized Jacobi structures, and thus the corresponding homology operators for free. For classical Jacobi structures we end up in this way with Lichnerowicz–Jacobi homology and generating operators for associated Lie algebroids (cf [LLMP, ILMP, Uch, Va1]).

Section 4 is devoted to Courant–Jacobi brackets, i.e. 'Jacobi versions' of Courant brackets and Courant algebroids (cf [Co, LWX, Ro, Wa, IM2]). This concept, again, is developed naturally by the method of gauging.

The last section contains pure algebraic generalizations of Jacobi algebroids. We start with a graded associative commutative algebra $\mathcal{A}$ and construct the graded Jacobi algebra $\mathcal{D}^\alpha(\mathcal{A})$ of first-order polydifferential operators on $\mathcal{A}$. The abstract Schouten–Jacobi bracket on $\mathcal{D}^\alpha(\mathcal{A})$ recognizes supercanonical elements (Jacobi structures) which generate graded Jacobi brackets on $\mathcal{A}$, so we can consider the corresponding cohomology operators as being Hamiltonian 'vector fields'. This can be viewed as a 'Jacobi version' of the results of Krasil'shchik [Kr2] and a graded algebraic generalization of this kind of structure described in [GM]. An abstract version of Lichnerowicz–Jacobi cohomology is defined.

2. Linear and affine Jacobi structures

Suppose that we are given a Poisson tensor Λ on a manifold N, which is homogeneous with respect to a vector field Δ (cf [DLM]), i.e. $[\Delta, \Lambda] = -\Lambda$, where $[\cdot, \cdot]$ stands for the Schouten bracket in the form which gives a graded Lie algebra structure on the graded space $A(M) = \oplus_i A^i(M)$ of multivector fields on M (this differs by a sign from the traditional Schouten bracket).

Lemma 1. *The pair $J = (\Lambda + \Gamma \wedge \Delta, \Gamma)$ is a Jacobi structure if and only if*

$$\Gamma \wedge [\Delta, \Gamma] \wedge \Delta = [\Gamma, \Lambda] \wedge \Delta. \qquad (12)$$

Proof. By direct calculations

$$[\Lambda + \Gamma \wedge \Delta, \Lambda + \Gamma \wedge \Delta] = [\Lambda, \Lambda] - 2[\Gamma, \Lambda] \wedge \Delta + 2\Gamma \wedge [\Delta, \Lambda] + 2\Gamma \wedge [\Delta, \Gamma] \wedge \Delta. \qquad (13)$$

Since $[\Lambda, \Lambda] = 0$ and $[\Delta, \Lambda] = -\Lambda$, this equals $-2\Gamma \wedge (\Lambda + \Gamma \wedge \Delta) = -2\Gamma \wedge \Lambda$ if and only if (12) is satisfied. $\qquad\square$

Corollary 1. *If Γ is a homogeneous vector field with $[\Gamma, \Lambda] = 0$, then $J = (\Lambda', \Gamma)$ with $\Lambda' = \Lambda + \Gamma \wedge \Delta$ is a Jacobi structure.*

Proof. Γ is homogeneous of degree k means that $[\Delta, \Gamma] = k\Gamma$ for certain k. Thus, $\Gamma \wedge [\Delta, \Gamma] = 0$ and the corollary follows by the lemma. $\qquad\square$

Corollary 2. *If f is a homogeneous function of degree k, then $J = (\Lambda', \Gamma)$ with $\Gamma = \Lambda_f$, $\Lambda' = \Lambda + \Lambda_f \wedge \Delta$ and $\Lambda_f = i_{df}\Lambda$ being the Hamiltonian vector field associated with f, is a Jacobi structure. Moreover, if $k \neq 0$, then this structure is tangent (i.e. Λ and Γ are tangent) to the submanifold determined by the equation $f = \frac{1}{k}$ (assuming that $\frac{1}{k}$ is a regular value of f).*

Proof. It is easy to see that the Hamiltonian vector field Λ_f is homogeneous of degree $(k-1)$. Then, J is a Jacobi structure due to the previous corollary. The function $f - \frac{1}{k}$ acts by the Jacobi bracket by

$$\left\{ f - \frac{1}{k}, \cdot \right\}_J = \Lambda_f - kf\Lambda_f + \left(f - \frac{1}{k} \right)\Lambda_f = \left(f - \frac{1}{k} \right)(1 - k)\Lambda_f \quad (14)$$

which vanishes on the described submanifold. $\qquad\square$

Note that similar observations have been done in [Pe]. Many important examples of Jacobi manifolds are of the above form, for instance, contact submanifolds of exact symplectic ones or spheres in duals of Lie algebras. A standard example of a homogeneous Poisson tensor is a linear Poisson tensor on a vector bundle E over a base manifold M with Δ being the Liouville vector field on E. Linearity means that the Poisson bracket of linear (along fibres) functions on E is again a linear function. Since every linear function is represented by a section of the dual bundle E^* by contraction, this gives a Lie bracket on sections of E^* which is a Lie algebroid bracket giving rise to the corresponding Schouten bracket on the graded space $A(E^*) = \oplus_i A^i(E^*)$ of multisections of E^*. A generalized version of this kind of bracket defined in [IM1] has been recognized in [GM] as a (graded) Jacobi bracket of degree -1 on $A(E^*)$. Being of degree -1, it defines a Lie bracket on a graded subspace $\mathrm{Aff}(E) = C^\infty(M) \oplus \Gamma(E^*)$ which determines the whole Jacobi bracket completely, due to being a first-order operator. The notation $\mathrm{Aff}(E)$ is justified by the fact that $\mathrm{Aff}(E)$ is just the graded space of affine functions on E with the obvious identification of $C^\infty(M)$ with the algebra of basic functions on E. The graded bracket on $\mathrm{Aff}(E)$ comes from a homogeneous Jacobi bracket on E determined by the Jacobi structure $J = (\Lambda + \Phi^v \wedge \Delta, \Phi^v)$, where Λ is a linear Poisson tensor and Φ^v is the vertical lift of a section Φ of E, which is a cocycle $d_\Lambda \Phi = 0$ with respect to the exterior derivative of the Lie algebroid associated with Λ (cf [IM1]). The cocycle property gives that $[\Phi^v, \Lambda] = 0$ and Φ^v is clearly homogeneous, so that this is precisely the kind of Jacobi structure described in corollary 1. This justifies the name Jacobi algebroid given to the bracket on $A(E^*)$ in [GM]. We can slightly generalize the result of [IM1] by considering arbitrary Jacobi structures which are linear with respect to a vector field Δ, i.e. which determine a Lie bracket on linear functions (homogeneous of degree 1). For functions, vector fields, etc, on a vector bundle E over M, a *homogeneous part* is defined. For a function f let k be the maximal number such that all vertical derivatives of order k vanish on M (identified with the zero-section). Then the homogeneous part f_0 is the homogeneous polynomial of order k such that all vertical derivatives of $f - f_0$ of order $(k + 1)$ vanish on M. For example, if the function f does not vanish on M then its homogeneous part is just the

pull-back of the function f restricted to M. The homogeneous part of a homogeneous function is just this function. For a vector field Γ, written in local coordinates near the zero-section by

$$\Gamma = f_i \partial_{y_i} + g_a \partial_{x^a} \tag{15}$$

where y_i are vertical coordinates and x^a are coordinates on the manifold M, the homogeneous part Γ_0 of Γ is just the first non-trivial homogeneous vector field $\Gamma_0 = f_i' \partial_{y_i}$ with f_i' being homogeneous of degree k such that the vertical derivatives of the vertical coordinates of $\Gamma - \Gamma_0$ vanish up to order $(k + 1)$. In particular, if Γ does not vanish on M, then $\Gamma_0 = f_i' \partial_{y_i}$, where f_i' is the pull-back of f_i restricted to M.

Theorem 1. *Every linear Jacobi structure $J = (\Lambda', \Gamma)$ on a vector bundle E induces a linear Poisson structure $\Lambda = \Lambda' - \Gamma \wedge \Delta$ such that $[\Gamma, \Lambda] = \Gamma \wedge [\Delta, \Gamma]$. It induces also a Jacobi structure $J' = (\Lambda + \Gamma_0 \wedge \Delta, \Gamma_0)$ with Γ_0 being the homogeneous part of Γ.*

Proof. It is easy to see that the bracket induced by the bivector $\Lambda = \Lambda' - \Gamma \wedge \Delta$ on linear functions coincides with the Jacobi bracket, i.e. the bracket is linear and the tensor Λ is homogeneous of degree -1. Thus,

$$[\Delta, \Lambda] = [\Delta, \Lambda'] - [\Delta, \Gamma] \wedge \Delta = -\Lambda = -\Lambda' + \Gamma \wedge \Delta. \tag{16}$$

We then obtain

$$[\Lambda, \Lambda] = [\Lambda', \Lambda'] + 2[\Gamma, \Lambda'] - 2\Gamma \wedge [\Delta, \Lambda'] + 2\Gamma \wedge [\Delta, \Gamma] \wedge \Delta = 0 \tag{17}$$

due to (16) and $[\Gamma, \Lambda'] = 0$, $[\Lambda', \Lambda'] = -2\Gamma \wedge \Lambda'$, so that Λ is a Poisson tensor. We have additionally

$$[\Gamma, \Lambda] = [\Gamma, \Lambda'] - \Gamma \wedge [\Gamma, \Delta] = [\Delta, \Gamma]. \tag{18}$$

Let now Γ_0 be the homogeneous part of Γ. For simplicity, assume that Γ_0 is of degree -1 (the general case can be proved in a completely analogous way). Put $\Gamma' = \Gamma - \Gamma_0$. We have then, according to (18),

$$[\Gamma' + \Gamma_0, \Lambda] = (\Gamma' + \Gamma_0) \wedge ([\Delta, \Gamma'] - \Gamma_0) \tag{19}$$

$$= \Gamma' \wedge [\Delta, \Gamma'] - \Gamma' \wedge \Gamma_0 + \Gamma_0 \wedge [\Delta, \Gamma']. \tag{20}$$

It is easy to see that the right-hand side vanishes on M when applied to a pair of linear functions. Since the same is true for $[\Gamma', \Lambda]$, also $[\Gamma_0, \Lambda]$ vanishes on M when applied to a pair of linear functions. But $[\Gamma_0, \Lambda]$ is a vertical tensor which is constant along fibres, so that $[\Gamma_0, \Lambda] = 0$. $\square$

Theorem 2. *Every Jacobi structure on a vector bundle E over M which is linear and affine (i.e. such that the linear functions and the affine functions are closed with respect to the Jacobi bracket) is of the form $J = (\Lambda + \Gamma \wedge \Delta, \Gamma)$, where Λ is a linear Poisson tensor and $\Gamma = \Gamma_0 + \Gamma_1$ is an affine vector field with the decomposition into homogeneous parts Γ_0, Γ_1 of orders -1 and 0, respectively, such that $[\Gamma_0, \Lambda] = 0$ and $[\Gamma_1, \Lambda] = \Gamma_0 \wedge \Gamma_1$. The vertical vector field Γ_0 is the vertical lift Φ^v of certain section Φ of E which is closed with respect to the exterior derivative associated with the Lie algebroid structure on E^* induced by Λ. If, additionally, the Jacobi bracket of a linear and a basic function is basic (i.e. the Jacobi bracket is homogeneous of degree -1), then $\Gamma_1 = 0$ and*

$$J = (\Lambda + \Delta \wedge \Phi^v, -\Phi^v) \tag{21}$$

with $d_\Lambda \Phi = 0$.

Proof. Since $J = (\Lambda', \Gamma)$ is affine, Γ is an affine vector field splitting into homogeneous parts $\Gamma = \Gamma_0 + \Gamma_1$. According to theorem 1, $\Lambda = \Lambda' - \Gamma \wedge \Delta$ is linear and $[\Gamma, \Lambda] = [\Gamma_0, \Lambda] + [\Gamma_1, \Lambda]$ equals

$$(\Gamma_0 + \Gamma_1) \wedge [\Delta, \Gamma_0 + \Gamma_1] = \Gamma_0 \wedge \Gamma_1. \tag{22}$$

Comparing the homogeneous parts of orders -1 and 0 we get $[\Gamma_0, \Lambda] = 0$ and $[\Gamma_1, \Lambda] = \Gamma_0 \wedge \Gamma_1$. $\qquad\square$

The homogeneous Jacobi structures have been studied in [IM]. The bracket corresponding to (21) has the form

$$\{f, g\}_J = \{f, g\} + (\Delta \wedge \Phi^v)(f, g) - f\Phi^v(g) + \Phi^v(f)g \tag{23}$$

where $\{\cdot, \cdot\}$ is the linear Poisson bracket associated with Λ. It is interesting that the bracket

$$[\![X, Y]\!]^\Phi = [\![X, Y]\!] + (\mathrm{Deg} \wedge i_\Phi)(X, Y) - X \wedge i_\Phi Y + (-1)^x i_\Phi X \wedge Y \tag{24}$$

has formally the same form with Deg playing the role of the Liouville vector field and i_φ being a graded derivative (vector field) of degree -1 with respect to Deg. We will show later that this is not accidental.

3. Jacobi algebroids and homology

Recall that for a vector bundle E over the base manifold M, we denote by $A(E) = \oplus_{k \in \mathbb{Z}} A^k(E)$ the exterior algebra of multisections of E. We will refer to elements of $A^k(E^*)$ as to *k-forms* on E. Here we identify $A^0(E)$ with the algebra $C^\infty(M)$ of smooth functions on the base and $A^k(E) = \{0\}$ for $k < 0$. Denote by $|X|$ the Grassmann degree of the multisection $X \in A(E)$. We will use the convention that we write x for $|X| - 1$—the shifted degree of X (this is the Lie algebra degree of X with respect to the Schouten bracket $[\![\cdot, \cdot]\!]$ induced by any Lie algebroid bracket on E).

This is the idea going back to Witten [Wi] to deform the de Rham exterior derivative by gauging the cotangent bundle by the multiplication by the function e^f:

$$\mathrm{d}^{\mathrm{d}f}\mu = \mathrm{e}^{-f}\,\mathrm{d}(\mathrm{e}^f\mu) = \mathrm{d}\mu + \mathrm{d}f \wedge \mu. \tag{25}$$

We have clearly $(\mathrm{d}^{\mathrm{d}f})^2 = 0$ and get the corresponding cohomology being equivalent to de Rham cohomology. This time, however, $\mathrm{d}^{\mathrm{d}f}$ is not a derivation but a first-order differential operator with respect to the wedge product on differential forms.

A natural generalization is to start with the exterior derivative d associated with a Lie algebroid structure in a vector bundle E over M and to take any 1-cocycle Φ instead of the coboundary $\mathrm{d}f$, so that $\mathrm{d}^\Phi\mu = \mathrm{d}\mu + \Phi \wedge \mu$. This is exactly the exterior differential we obtain for a Jacobi algebroid associated with the 1-cocycle Φ by an analogue of the Cartan formula (cf [IM1, GM])

$$\mathrm{d}^\Phi\mu(X_1, \ldots, X_{k+1}) = \sum_i (-1)^{i+1} [\![X_i, \mu(X_1, \ldots, \hat{X}_i, \ldots, X_{k+1})]\!]^\Phi$$

$$+ \sum_{i<j} (-1)^{i+j} \mu([\![X_i, X_j]\!]^\Phi, X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{k+1}). \tag{26}$$

Even if the 1-cocycle Φ is not exact, there is a nice construction [IM1] which allows us to view Φ as being exact but for an extended Lie algebroid in the bundle $\hat{E} = E \times \mathbb{R}$ over $M \times \mathbb{R}$. The sections of this bundle may be viewed as time-dependent sections of E. The sections of E form a Lie subalgebra of time-independent sections in the Lie algebroid $\hat{E}$ which generate the $C^\infty(M \times \mathbb{R})$-module of sections of $\hat{E}$ and the whole structure is uniquely determined by

putting the anchor $\hat{\rho}(X)$ of a time-independent section X to be $\hat{\rho}(X) = \rho(X) + \langle \Phi, X \rangle \partial_t$, where t is the standard coordinate function in $\mathbb{R}$ and ρ is the anchor in E. All this is consistent (thanks to the fact that Φ is a cocycle) and defines a Lie algebroid structure on $\hat{E}$ with the exterior derivative d satisfying $dt = \Phi$.

Let now $U : A(E) \to A(\hat{E})$ be a natural embedding of the Grassmann algebra of E into the Grassmann subalgebra of time-independent sections of $\hat{E}$. It is obvious that U is a homomorphism of the corresponding Schouten brackets

$$[\![U(X), U(Y)]\!]\hat{} = U([\![X, Y]\!]) \tag{27}$$

where we use the notation $[\![\cdot, \cdot]\!]$ and $[\![\cdot, \cdot]\!]\hat{}$ for the Schouten brackets in E and $\hat{E}$, respectively. Let us now gauge $A(E)$ inside $A(\hat{E})$ by putting

$$\tilde{U}(X) = e^{-xt} U(X) \tag{28}$$

for any homogeneous element X. Note that $\tilde{U}$ preserves the grading but not the wedge product. It can easily be proved (cf [GM] where this is proved for an extension of $\hat{E}$) that the Jacobi algebroid bracket (7) can be obtained by this gauging $A(E)$ in $A(\hat{E})$.

Theorem 3. *For any homogeneous elements $X, Y \in A(E)$ we have*

$$[\![\tilde{U}(X), \tilde{U}(Y)]\!]\hat{} = \tilde{U}([\![X, Y]\!] + xX \wedge i_\Phi Y - (-1)^x y i_\Phi X \wedge Y) = \tilde{U}([\![X, Y]\!]^\Phi). \tag{29}$$

From the above theorem we get for free the following.

Corollary 3 ([IM1]). *The Schouten–Jacobi bracket (7) is a graded Lie bracket for $A(E)$.*

Thus we have obtained the Jacobi algebroid bracket and the corresponding exterior differential by gauging. What about the other ingredients of the Cartan calculus? The contraction is obvious, so let us define the Lie differential. For a Lie algebroid E the Lie differential $\mathcal{L}_X$ along a multisection X of E acting on $A(E^*)$ is defined by

$$\mathcal{L}_X \mu = i_X d\mu + (-1)^x di_X \mu. \tag{30}$$

We have the following well-known formulae (cf, e.g., [KSM, Mi])

$$[\![\mathcal{L}_X, \mathcal{L}_Y]\!] = -\mathcal{L}_{[\![Y,X]\!]} \tag{31}$$

$$[\![\mathcal{L}_X, i_Y]\!] = -i_{[\![Y,X]\!]} \tag{32}$$

where

$$[\![\mathcal{L}_X, \mathcal{L}_Y]\!] = \mathcal{L}_X \circ \mathcal{L}_Y - (-1)^{xy} \mathcal{L}_Y \circ \mathcal{L}_X \tag{33}$$

$$[\![\mathcal{L}_X, i_Y]\!] = \mathcal{L}_X \circ i_Y - (-1)^{(y+1)x} i_Y \circ \mathcal{L}_X \tag{34}$$

are the graded commutators of the graded morphism of $A(E^*)$. If we define the deformed Lie differential by

$$\mathcal{L}_X^\Phi \mu = i_X d^\Phi \mu + (-1)^x d^\Phi i_X \mu \tag{35}$$

then obviously

$$[\![\mathcal{L}_X^\Phi, d^\Phi]\!] = \mathcal{L}_X^\Phi \circ d^\Phi - (-1)^x d^\Phi \circ \mathcal{L}_X^\Phi = 0. \tag{36}$$

Using the formula

$$i_X(\Phi \wedge \mu) = i_{X_\Phi} \mu - (-1)^x \Phi \wedge i_X \mu \tag{37}$$

where we write X_Φ for $i_\Phi X$, we obtain

$$\mathcal{L}_X^\Phi = \mathcal{L}_X + i_{X_\Phi}. \tag{38}$$

Note that this coincides with the definition of Φ-Lie derivative in [IM1] for X being just sections of E. In this case $i_{X_\Phi}\mu = \langle X, \Phi\rangle\mu$. However, in spite of the fact that the Lie differential was deformed, we get the same formulae as (31) with the original Schouten bracket instead of the Schouten–Jacobi bracket.

Theorem 4. *The following identities hold:*

$$[\![\mathcal{L}_X^\Phi, \mathcal{L}_Y^\Phi]\!] = -\mathcal{L}_{[\![Y,X]\!]}^\Phi \qquad (39)$$

$$[\![\mathcal{L}_X^\Phi, i_Y]\!] = -i_{[\![Y,X]\!]}. \qquad (40)$$

Proof. Since $[\![i_X, i_Y]\!] = 0$, we obtain

$$[\![\mathcal{L}_X^\Phi, i_Y]\!] = [\![\mathcal{L}_X + i_{X_\Phi}, i_Y]\!] = -i_{[\![Y,X]\!]}. \qquad (41)$$

Now,

$$[\![\mathcal{L}_X^\Phi, \mathcal{L}_Y^\Phi]\!] = [\![\mathcal{L}_X^\Phi, [\![i_Y, d^\Phi]\!]]\!] = -[\![i_{[\![Y,X]\!]}, d^\Phi]\!] = -\mathcal{L}_{[\![Y,X]\!]}^\Phi \qquad (42)$$

since $\mathcal{L}_X^\Phi = [\![i_X, d^\Phi]\!]$ and $[\![\mathcal{L}_X^\Phi, d^\Phi]\!] = 0$. $\qquad\square$

To obtain similar formulae but with the Schouten–Jacobi bracket we will again use a proper gauging. Let us use the embedding $\hat{U}$ of the Grassmann algebra $A(E^*)$ into $A(\hat{E}^*) = A((\hat{E})^*)$ by $\hat{U}_a(\mu) = e^{(|\mu|+a)t}U(\mu)$. Here the elements $U(\mu)$ are regarded as time-independent sections. Using the standard Lie differential $\mathcal{L}$ and the exterior derivative d for the extended Lie algebroid $\hat{E}$, we get the following.

Lemma 2.

$$i_{\tilde{U}(X)}\hat{U}_a(\mu) = e^t\hat{U}_a(i_X\mu) \qquad (43)$$

$$\mathcal{L}_{\tilde{U}(X)}\hat{U}_a(\mu) = \hat{U}_a\left(\mathcal{L}_X\mu + (|\mu|+a)i_{X_\Phi}\mu - (-1)^x x\Phi\wedge i_X\mu\right). \qquad (44)$$

Proof. We will write, for simplicity, X and μ instead of $U(X)$ and $U(\mu)$, regarding $A(E)$ and $A(E^*)$ as time-independent sections in $A(\hat{E})$ and $A(\hat{E}^*)$, respectively. The Schouten–Nijenhuis bracket on embedded $A(E)$ and the exterior derivative on $A(E^*)$ coincide with the restrictions of the Schouten bracket on $A(\hat{E})$ and the exterior derivative on $A(\hat{E}^*)$, respectively. We have clearly

$$i_{\tilde{U}(X)}\hat{U}_a(\mu) = e^{-xt}i_X(e^{(|\mu|+a)t}\mu) = e^t\hat{U}_a(i_X\mu). \qquad (45)$$

Using the fact that $dt = \Phi$, we get in turn

$$\begin{aligned}\mathcal{L}_{\tilde{U}(X)}\hat{U}_a(\mu) &= i_{\tilde{U}(X)}d(e^{(|\mu|+a)t}\mu) + (-1)^x di_{\tilde{U}(X)}(e^{(|\mu|+a)t}\mu)\\ &= e^{-xt}i_X((|\mu|+a)e^{(|\mu|+a)t}\Phi\wedge\mu + e^{(|\mu|+a)t}d\mu) + (-1)^x d(e^{(|\mu|+a-x)t}i_X\mu)\\ &= \hat{U}_a\left(\mathcal{L}_X\mu + (|\mu|+a)i_{X_\Phi}\mu - (-1)^x x\Phi\wedge i_X\mu\right).\end{aligned} \qquad (46)$$

$\qquad\square$

For the Jacobi algebroid associated with the 1-cocycle Φ we will call

$$\pounds_X^{\Phi,a}\mu = \mathcal{L}_X\mu + (|\mu|+a)i_{X_\Phi}\mu - (-1)^x x\Phi\wedge i_x\mu \qquad (47)$$

the *Jacobi–Lie differential* of μ along X. According to the above lemma,

$$\mathcal{L}_{\tilde{U}(X)}\hat{U}_a(\mu) = \hat{U}_a\left(\pounds_X^{\Phi,a}\mu\right) \qquad (48)$$

and we have the following analogue of (39).

Theorem 5.

$$[\![\pounds_X^{\Phi,a}, \pounds_Y^{\Phi,a}]\!] = -\pounds_{[\![Y,X]\!]^\Phi}^{\Phi,a}. \qquad (49)$$

Proof. In view of (48),

$$\hat{U}_a\big(\pounds_X^{\Phi,a}\circ\pounds_Y^{\Phi,a}(\mu)\big)=\mathcal{L}_{\tilde{U}(X)}\circ\mathcal{L}_{\tilde{U}(Y)}\hat{U}_a(\mu) \tag{50}$$

so that

$$\hat{U}_a\big(\llbracket\pounds_X^{\Phi,a},\pounds_Y^{\Phi,a}\rrbracket(\mu)\big)=\llbracket\mathcal{L}_{\tilde{U}(X)},\mathcal{L}_{\tilde{U}(Y)}\rrbracket\hat{U}_a(\mu)=-\mathcal{L}_{\llbracket\tilde{U}(Y),\tilde{U}(X)\rrbracket}\hat{U}_a(\mu) \tag{51}$$

$$=-\mathcal{L}_{\tilde{U}(\llbracket Y,X\rrbracket^{\Phi})}\hat{U}_a(\mu)=\hat{U}_a\big(-\pounds_{\llbracket Y,X\rrbracket^{\Phi}}^{\Phi,a}(\mu)\big) \tag{52}$$

and the theorem follows, since $\hat{U}$ is injective. $\square$

Instead of (40) we have the following deformed version.

Theorem 6.

$$\llbracket\pounds_X^{\Phi,a},i_Y\rrbracket=-i_{\llbracket Y,X\rrbracket^{\Phi}}-(-1)^x i_{Y\wedge X_{\Phi}}. \tag{53}$$

Proof. Analogously as above we get by (43)

$$\llbracket\mathcal{L}_{\tilde{U}(X)},\mathrm{e}^{-t}i_{\tilde{U}(Y)}\rrbracket\hat{U}_a(\mu)=\hat{U}_a\big(\llbracket\pounds_X^{\Phi,a},i_Y\rrbracket(\mu)\big). \tag{54}$$

Now, using

$$\mathcal{L}_Z(\mathrm{e}^{-t}\mu)=\mathrm{e}^{-t}\big(\mathcal{L}_Z\mu+(-1)^z i_{Z_{\Phi}}\mu\big) \tag{55}$$

we get

$$\hat{U}_a\big(\llbracket\pounds_X^{\Phi,a},i_Y\rrbracket(\mu)\big)=\mathrm{e}^{-t}\big(\llbracket\mathcal{L}_{\tilde{U}(X)},i_{\tilde{U}(Y)}\rrbracket+(-1)^x i_{\tilde{U}(X)_{\Phi}}\circ i_{\tilde{U}(Y)}\big)\hat{U}_a(\mu) \tag{56}$$

$$=\mathrm{e}^{-t}\big(-i_{\llbracket\tilde{U}(Y),\tilde{U}(X)\rrbracket}+(-1)^x\,\mathrm{e}^{-t}i_{\tilde{U}(X_{\Phi})}\circ i_{\tilde{U}(Y)}\big)\hat{U}_a(\mu) \tag{57}$$

$$=\hat{U}_a\big(-i_{\llbracket Y,X\rrbracket^{\Phi}}-(-1)^x i_{Y\wedge X_{\Phi}}\big) \tag{58}$$

and the theorem follows. $\square$

An element $P\in A^2(E)$ with $\llbracket P,P\rrbracket^{\Phi}=0$ we will call a *Jacobi element* (or *canonical structure*). An immediate consequence of (44) is the following observation.

Corollary 4. *For a Jacobi element P of $A(E)$ the Lie differential*

$$\pounds_P^{\Phi,a}\mu=\mathcal{L}_P\mu+(|\mu|+a)i_{P_{\Phi}}+\Phi\wedge i_P\mu \tag{59}$$

is a homology operator on $A(E^*)$, i.e. $|\pounds_P^{\Phi,a}\mu|=|\mu|-1$ and $\big(\pounds_P^{\Phi,a}\big)^2=0$. Moreover, $\pounds_P^{\Phi,a}$ is a generating operator for the Schouten–Nijenhuis bracket on $A(E^*)$:

$$\llbracket\mu,\nu\rrbracket_P=(-1)^{|\mu|}\big(\pounds_P^{\Phi,a}(\mu\wedge\nu)-\pounds_P^{\Phi,a}(\mu)\wedge\nu-(-1)^{|\mu|}\mu\wedge\pounds_P^{\Phi,a}(\nu)\big) \tag{60}$$

which does not depend on a and which is the Schouten–Nijenhuis bracket of the Lie algebroid bracket on E^*, defined for $\mu,\nu\in A^1(E^*)$ by

$$\llbracket\mu,\nu\rrbracket_P=i_{P_{\mu}}\mathrm{d}^{\Phi}\nu-i_{P_{\nu}}\mathrm{d}^{\Phi}\mu+\mathrm{d}^{\Phi}\langle P,\mu\wedge\nu\rangle \tag{61}$$

$$=\mathcal{L}_{P_{\mu}}^{\Phi}\nu-\mathcal{L}_{P_{\nu}}^{\Phi}\mu-\mathrm{d}^{\Phi}\langle P,\mu\wedge\nu\rangle. \tag{62}$$

Since $\pounds_P^{\Phi,a}-\pounds_P^{\Phi,a'}=(a-a')i_{P_{\Phi}}$ is a derivation, all Lie differentials $\pounds_P^{\Phi,a}$ are equally good as generators of the Schouten–Nijenhuis bracket (59). In [ILMP], theorem 4.8, the authors have introduced $\pounds_P^{\Phi,0}$ and $\pounds_P^{\Phi,1}$. The first Lie differential $\pounds_X^{\Phi,0}$ (denoted simply by $\pounds_X^{\Phi}$) reduces to $\mathcal{L}_X^{\Phi}$ on 1-forms for X being just a section of E. The second Lie differential $\pounds_X^{\Phi,1}$, applied

for functions, describes, for $X \in A^1(E)$, the bracket: $\pounds_X^{\Phi,1}(f) = [\![X, f]\!]^\Phi$. The homology defined by $\pounds_P^\Phi$ for a Jacobi element P we will call Lichnerowicz–Jacobi homology and denote by $H_*^{LJ}(E, P)$. The homology defined by $\pounds_P^{\Phi,1}$ we will call Jacobi homology and denote by $H_*^J(E, P)$.

The fact that $\left(\pounds_P^{\Phi,a}\right)^2 = 0$ for all a has the obvious consequence that $\pounds_P^\Phi$ and i_{P_Φ} commute.

Theorem 7. *For a Jacobi element P*

$$[\![\pounds_P^\Phi, i_{P_\Phi}]\!] = \pounds_P^\Phi \circ i_{P_\Phi} + i_{P_\Phi} \circ \pounds_P^\Phi = 0. \tag{63}$$

Remark. The bracket (61) has been introduced in [IM1] as a generalization of the triangular Lie bialgebroid [MX] for Jacobi algebroids (generalized Lie algebroids), and it is an obvious generalization of the Koszul–Fuchssteiner [Kz, Fu] bracket on 1-forms induced by a Poisson structure.

Example 1. If $E = TM \oplus \mathbb{R}$ is a Lie algebroid of first-order differential operators on $C^\infty(M)$, i.e. the Lie bracket on sections reads

$$[\![(X, f), (Y, g)]\!] = ([X, Y], X(g) - Y(f)) \tag{64}$$

and the 1-cocycle is $\Phi((X, f)) = f$, then every Jacobi element $P = (\Lambda, \Gamma)$ being a section of $A^2(E) = \wedge^2 TM \oplus TM$ is just a Jacobi structure on M. Identifying elements of $\Omega^k(E)$ with pairs (μ, ν), where μ is a k-form on M and ν is a $(k-1)$-form on M, we get $d(\mu, \nu) = (d\mu, -d\nu)$ and

$$\mathcal{L}_{(\Lambda,\Gamma)}(\mu, \nu) = (\mathcal{L}_\Lambda \mu - \mathcal{L}_\Gamma \nu, -\mathcal{L}_\Lambda \nu). \tag{65}$$

Thus,

$$\pounds_P^\Phi(\mu, \nu) = (\mathcal{L}_\Lambda \mu - \mathcal{L}_\Gamma \nu + k i_\Gamma \mu, -\mathcal{L}_\Lambda \nu + i_\Lambda \mu - (k-1) i_\Gamma \nu) \tag{66}$$

so we get exactly the standard Lichnerowicz–Jacobi homology operator as described in [LLMP, ILMP, Va1, BMMP]. For a Poisson structure it reduces to the Koszul–Brylinski boundary operator (see, e.g., [Br] or [Va], chapter 5). The homology operator $\pounds_P^{\Phi,1}$ is related to Chevalley–Eilenberg homology operator δ^{CE} of the corresponding Jacobi bracket with respect to the maps

$$\pi_k : C^\infty(M) \times \Lambda^k C^\infty(M) \to \Omega^k(E) \qquad \pi_k(f \times f_1 \wedge \ldots \wedge f_k) = f \, d^\Phi f_1 \wedge \ldots \wedge d^\Phi f_k \tag{67}$$

by $\pounds_P^{\Phi,1} \circ \pi_k = \pi_{k-1} \circ \delta^{CE}$.

4. Courant–Jacobi algebroids

The method of gauging can be used to deform the *Courant brackets* associated with a Lie algebroid bracket $[\cdot, \cdot]$ on E. Recall that the Courant bracket [Co] is the following bracket on $E \oplus E^*$:

$$[X + \xi, Y + \eta]_C = [X, Y] + \mathcal{L}_X \eta - \mathcal{L}_Y \xi + \tfrac{1}{2} d(i_Y \xi - i_X \eta). \tag{68}$$

Dirac structures, introduced independently by Courant and Weinstein [CW] and Dorfman [Do1] for $E = TM$, can be defined as subbundles L of $E \oplus E^*$ which are maximally isotropic under the canonical symmetric pairing $\langle X + \xi, Y + \eta \rangle_+ = i_Y \xi + i_X \eta$ and closed with respect to the Courant bracket. For their use in studying completely integrable systems of partial differential equations we refer to [Do2].

Now, choosing a 1-cocycle $\Phi \in \Omega^1(E)$, we can consider the extended Lie algebroid $\hat{E}$ defined in the previous section and the embedding

$$\mathbf{U}(X + \xi) = U(X) + \hat{U}_0(\xi) \tag{69}$$

of $\mathrm{Sec}(E \oplus E^*)$ into $\mathrm{Sec}(\hat{E} \oplus \hat{E}^*)$. In other words, $\mathbf{U}(X + \xi) = X + \mathrm{e}^t \xi$ when X, ξ are regarded as time-independent sections of $\hat{E}$ and $\hat{E}^*$, respectively. For computational aims we can just think that this is a true gauging, i.e. we work in E and E^* and $t \in C^\infty(M)$ is the potential for Φ, i.e. $\mathrm{d}t = \Phi$.

In $\hat{E} \oplus \hat{E}^* = \hat{E} \oplus \widehat{E}^*$ we have its own Courant bracket $[\cdot, \cdot]_C^\wedge$. Now, following the ideas of the previous sections, one proves easily that

$$[\mathbf{U}(X + \xi), \mathbf{U}(Y + \eta)]_C^\wedge = \mathbf{U}\left([X, Y] + \mathcal{L}_X^\Phi \eta - \mathcal{L}_Y^\Phi \xi + \tfrac{1}{2}\,\mathrm{d}^\Phi(i_Y \xi - i_X \eta)\right). \tag{70}$$

The bracket

$$[X + \xi, Y + \eta]_C^\Phi = [X, Y] + \mathcal{L}_X^\Phi \eta - \mathcal{L}_Y^\Phi \xi + \tfrac{1}{2}\,\mathrm{d}^\Phi(i_Y \xi - i_X \eta) \tag{71}$$

we will call the *Courant–Jacobi bracket* (associated with Φ). Similarly as above, a *Dirac–Jacobi structure* is a subbundle L of $E \oplus E^*$ which is maximally isotropic under the canonical symmetric pairing and closed with respect to the Courant–Jacobi bracket.

Example 2. Let $E = TM \oplus \mathbb{R}$ be the Lie algebroid of first-order differential operators on M. The sections of E are identified with pairs (X, f), where X is a vector field on M and $f \in C^\infty(M)$, and the Lie algebroid bracket reads

$$[(X_1, f_1), (X_2, f_2)] = ([X_1, X_2], X_1(f_2) - X_2(f_1)). \tag{72}$$

The sections of the dual bundle are pairs (α, h), where α is a 1-form on M and $h \in C^\infty(M)$, with the obvious pairing. The Lie algebroid exterior derivative of a function is $\mathrm{d}f = (\mathrm{d}f, 0)$, where d on the right-hand side is the standard exterior derivative (we hope that this abuse in notation will cause no confusion) and the Lie derivative is given by

$$\mathcal{L}_{(X,f)}(\alpha, h) = (\mathcal{L}_X \alpha + h\,\mathrm{d}f, X(h)). \tag{73}$$

Thus the Courant bracket on $E \oplus E^*$ reads

$$\begin{aligned}
[(X_1, f_1) + (\alpha_1, g_1), (X_2, f_2) + (\alpha_2, g_2)]_C =\ & ([X_1, X_2], X_1(f_2) - X_2(f_1)) \\
& + \big(\mathcal{L}_{X_1}\alpha_2 - \mathcal{L}_{X_2}\alpha_1 + g_2\,\mathrm{d}f_1 - g_1\,\mathrm{d}f_2 + \tfrac{1}{2}\mathrm{d}(i_{X_2}\alpha_1 - i_{X_1}\alpha_2 + f_2 g_1 - f_1 g_2), \\
& X_1(g_2) - X_2(g_1)\big).
\end{aligned} \tag{74}$$

The standard 1-cocycle Φ on E leading to the Schouten–Jacobi algebra of first-order polydifferential operators is given by $\Phi((X, f)) = f$. It is easy to see now that the corresponding Courant–Jacobi bracket is given by

$$\begin{aligned}
[(X_1, f_1) + (\alpha_1, g_1), (X_2, f_2) + (\alpha_2, g_2)]_C^\Phi =\ & ([X_1, X_2], X_1(f_2) - X_2(f_1)) \\
& + \big(\mathcal{L}_{X_1}\alpha_2 - \mathcal{L}_{X_2}\alpha_1 + g_2\,\mathrm{d}f_1 - g_1\,\mathrm{d}f_2 + \tfrac{1}{2}\mathrm{d}(i_{X_2}\alpha_1 - i_{X_1}\alpha_2 + f_2 g_1 - f_1 g_2) \\
& + f_1\alpha_2 - f_2\alpha_1, X_1(g_2) - X_2(g_1) + \tfrac{1}{2}(i_{X_2}\alpha_1 - i_{X_1}\alpha_2 - f_2 g_1 + f_1 g_2)\big).
\end{aligned} \tag{75}$$

This is exactly the bracket introduced by Wade [Wa] to define $\mathcal{E}^1(M)$-Dirac structures.

The Courant bracket (68) can be generalized to a bracket associated with a Lie bialgebroid

$$((E, [\cdot, \cdot]_E), (E^*, [\cdot, \cdot]_{E^*})) \tag{76}$$

in the sense of Mackenzie and Xu [MX] (which, in turn, is a fundamental example of a *Courant algebroid* bracket [LWX]):

$$[X + \xi, Y + \eta]_C = ([X, Y]_E + \mathcal{L}_\xi^{E^*} Y - \mathcal{L}_\eta^{E^*} X - \tfrac{1}{2} \, \mathrm{d}_{E^*}(i_Y \xi - i_X \eta) \tag{77}$$

$$+ \left([\xi, \eta]_{E^*} + \mathcal{L}_X^E \eta - \mathcal{L}_Y^E \xi + \tfrac{1}{2} \, \mathrm{d}_E(i_Y \xi - i_X \eta)\right). \tag{78}$$

Consider now a Jacobi bialgebroid [GM] (generalized Lie bialgebroid in the sense of [IM1])

$$\left(\left(E, [\cdot, \cdot]_E^{\Phi_0}\right), \left(E^*, [\cdot, \cdot]_{E^*}^{X_0}\right)\right) \tag{79}$$

associated with 1-cocycles $\Phi_0 \in \mathrm{Sec}(E^*)$, $X_0 \in \mathrm{Sec}(E)$ with respect to the Lie algebroid brackets on E and E^*, respectively. Let $\left(\hat{E}, [\cdot, \cdot]_{\hat{E}}^\wedge\right)$ and $\left(\hat{E}^*, [\cdot, \cdot]_{\hat{E}^*}^\wedge\right)$ be extended Lie algebroids associated with these cocycles with $\mathrm{d}_E(t) = \Phi_0$ and $\mathrm{d}_{E^*}(t) = X_0$. Let $[\cdot, \cdot]_{\hat{E}^*}^0$ be the Lie algebroid bracket on $\hat{E}^*$ obtained from $[\cdot, \cdot]_{\hat{E}^*}^\wedge$ by gauging by e^{-t}:

$$[\xi, \eta]_{\hat{E}^*}^0 = \mathrm{e}^t [\mathrm{e}^{-t} \xi, \mathrm{e}^{-t} \eta]_{\hat{E}^*}^\wedge. \tag{80}$$

It has been proved in [IM1], theorem 4.11, that

$$\left(\left(\hat{E}, [\cdot, \cdot]_{\hat{E}}^\wedge\right), \left(\hat{E}^*, [\cdot, \cdot]_{\hat{E}^*}^0\right)\right) \tag{81}$$

is a Lie bialgebroid, so we can consider the corresponding Courant algebroid bracket $[\cdot, \cdot]_C^\wedge$ (77) on $\hat{E} \oplus \hat{E}^*$. Now, using $\mathbf{U} : E \oplus E^* \to \hat{E} \oplus \hat{E}^*$, we get

$$[\mathbf{U}(X + \xi), \mathbf{U}(Y + \eta)]_C^\wedge = \mathbf{U}\left([X + \xi, Y + \eta]_C^{\Phi_0, X_0}\right) \tag{82}$$

where

$$[X + \xi, Y + \eta]_C^{\Phi_0, X_0} = \left([X, Y]_E + \mathcal{L}_\xi^{E^*} Y - \mathcal{L}_\eta^{E^*} X - \tfrac{1}{2} \, \mathrm{d}_{E^*}(i_Y \xi - i_X \eta) + \tfrac{1}{2}(i_Y \xi - i_X \eta) X_0\right)$$

$$+ \left([\xi, \eta]_{E^*} + (\mathcal{L}^E)_X^{\Phi_0} \eta - (\mathcal{L}^E)_Y^{\Phi_0} \xi + \tfrac{1}{2} \, \mathrm{d}_E^{\Phi_0}(i_Y \xi - i_X \eta)\right). \tag{83}$$

This is the *Courant–Jacobi bracket* associated with the Jacobi bialgebroid. In the case when $[\cdot, \cdot]_{E^*}$ is trivial and $X_0 = 0$ we end up with (71).

Being obtained by gauging this bracket has properties similar to those of Courant algebroid bracket. The abstract of these properties leads to the following definition (based on the definition of Courant algebroid proposed in [Ro] rather than on the original definition in [LWX]). We have reduced the number of axioms using ideas similar to [Uch].

Definition 1. *A Courant–Jacobi algebroid is a vector bundle F over M together with*

(i) *a nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ on the bundle,*

(ii) *a Loday operation $\circ$ on $\mathrm{Sec}(F)$ (i.e. a bilinear operation satisfying the Jacobi identity; the terminology goes back to [KS1]),*

(iii) *a bundle map $\rho : F \to TM \oplus \mathbb{R}$ which is a homomorphism into the Lie algebroid of first-order differential operators*

$$[\rho(e_1), \rho(e_2)] = \rho(e_1 \circ e_2), \tag{84}$$

satisfying the following properties:

1. *$\langle e_1 \circ e, e \rangle = \langle e_1, e \circ e \rangle$,*
2. *$\rho(e_1)(\langle e, e \rangle) = 2\langle e_1 \circ e, e \rangle$,*

for all $e, e_1, e_2 \in \mathrm{Sec}(F)$.

Note that we can reformulate the above definition in terms of the first-order differential operator

$$\mathcal{D} : C^\infty(M) \to Sec(F) \qquad \langle \mathcal{D}(f), e \rangle = \rho(e)(f). \qquad (85)$$

The Courant–Jacobi algebroid (like the Courant algebroid) is a particular case of a pure algebraic structure described in [JL], theorem 3.3. For a Jacobi bialgebroid (79) the corresponding Courant–Jacobi algebroid bracket on $F = E \oplus E^*$ is

$$(X + \xi) \circ (Y + \eta) = \left([X, Y]_E + \mathcal{L}_\xi^{E^*} Y - i_\eta \, \mathrm{d}_{E^*} X + \langle Y, \xi \rangle X_0 \right)$$
$$+ \left([\xi, \eta]_{E^*} + (\mathcal{L}^E)_X^{\Phi_0} \eta - i_Y \, \mathrm{d}_E^{\Phi_0} \xi \right). \qquad (86)$$

The symmetric form is $\langle \cdot, \cdot \rangle_+$, and $\rho = \rho_E + \rho_{E^*} + i_{\Phi_0 + X_0}$.

The definition of a *Dirac structure* associated with a Courant–Jacobi algebroid is obvious. We will postpone a study of Courant–Jacobi algebroids to a separate paper. Note only the following.

Theorem 8. *Every Dirac structure L associated with a Courant–Jacobi algebroid structure on E induces on L a Jacobi algebroid structure with the bracket being the restriction of $\circ$ to sections of L and the 1-cocycle being the restriction of $\langle \Phi, \cdot \rangle$ to sections of L.*

5. Krasil'shchik calculus for first-order differential operators

In this section, we will present the Krasil'shchik's approach to Schouten brackets [Kr1, Kr2] adapted to Jacobi structures, i.e. we will show that the identification of the concepts of Schouten and Poisson brackets can be extended to canonical brackets of first-order polydifferential operators and Jacobi brackets.

The differential calculus for associative commutative algebras has been developed by Vinogradov [ViA] (see also [KLV, VK, ViM]).

In [GM] we have observed that the Schouten bracket on multivector fields (i.e. elements of $A(TM)$) can be viewed as the restriction of the Richardson–Nijenhuis bracket on multilinear operators on $C^\infty(M)$ to polyderivations. At the same time, when reducing the Richardson–Nijenhuis bracket to first-order polydifferential operators, we get not a Schouten-type but a Jacobi-type bracket which is a particular case of what was recently studied in [IM1] under the name of a generalized Lie algebroid. Since a very general approach to Schouten brackets and supercanonical structures has been already developed in [Kr2], we will follow these ideas to develop a similar calculus 'on the Jacobi level'. We slightly change the approach of [Kr2] using rather a shift $\alpha \in \mathbb{Z}^n$ in the original grading than a divided grading. Of course, any such shift defines a divided grading by parity of coefficients of α so that the signs in the Krasil'shchik's and our approaches remain the same, but using the shift allows us to trace better the graded structures which are introduced and fits better to the concept of graded Poisson or Jacobi algebras. We will mostly skip the proofs which are standard inductions and just matters of simple calculations completely parallel to those in [Kr2]. All elements considered in formulae are uniform, i.e. with a well-defined degree.

We start with an n-graded associative commutative algebra $\mathcal{A}$ with unity **1** and will write simply a instead of $|a| \in \mathbb{Z}^n$. Let us fix $\alpha \in \mathbb{Z}^n$.

We define the graded $\mathcal{A}$-bimodules $\mathcal{D}_i^\alpha(\mathcal{A})$ (denoted shortly by $\mathcal{D}_i^\alpha$ if $\mathcal{A}$ is fixed) of genus i of polydifferential operators of first order by induction. Note that the genus relates the right-module structure to the left-module structure by

$$p \cdot a = (-1)^{\langle a, p + i\alpha \rangle} a \cdot p \qquad (87)$$

for $p \in \mathcal{D}_i^\alpha$, $a \in \mathcal{A}$, so that we get the right-module structure from the left one by definition. For $\mathcal{D}_0^\alpha$ we take just $\mathcal{A}$ (we can start with an arbitrary module of genus 0, but this choice is sufficient for our purposes in this paper). Then, we take $\mathcal{D}_1^\alpha$ as the space of those linear graded maps $D : \mathcal{A} \to \mathcal{A}$ which satisfy

$$D(ab) = D(a)b + (-1)^{\langle a,D \rangle} aD(b) - D(\mathbf{1})ab \tag{88}$$

i.e. $\mathcal{D}_1^\alpha$ is the module of first-order differential operators on $\mathcal{A}$. For $i > 1$ we define inductively $\mathcal{D}_i^\alpha$ as formed by those graded linear maps $D : \mathcal{A} \to \mathcal{D}_{i-1}^\alpha$ for which

$$D(ab) = D(a) \cdot b + (-1)^{\langle a,D+(i-1)\alpha \rangle} a \cdot D(b) - D(\mathbf{1}) \cdot ab \tag{89}$$

$$D(a,b) = -(-1)^{\langle a+\alpha,b+\alpha \rangle} D(b,a). \tag{90}$$

The notation is clearly

$$D(a_1,\ldots,a_j) = D(a_1,\ldots,a_{j-1})(a_j). \tag{91}$$

The left module structure on $\mathcal{D}_i^\alpha$ is obvious: $(a \cdot D)(b) = a \cdot D(b)$, the right one is determined by genus. There is a natural graded subspace

$$\mathcal{DE}^\alpha(\mathcal{A}) = \oplus_{i=0}^\infty \mathcal{DE}_i^\alpha(\mathcal{A}) \tag{92}$$

of polyderivations, i.e. $\mathcal{DE}_i^\alpha$ consists of those elements D from $\mathcal{D}_i^\alpha$ such that $D(\mathbf{1}) = 0$ (by graded symmetry this means that $D(\ldots,\mathbf{1},\ldots) = 0$).

As in the classical case, $\mathcal{D}_1^\alpha = \mathcal{DE}_1^\alpha \oplus \mathcal{A} \cdot I$, where $I \in \mathcal{D}_1^\alpha$ is the identity operator on $\mathcal{A}$. We will show a generalization of this fact later (see (112)). We can extend the product in the algebra $\mathcal{A}$ to the space

$$\mathcal{D}^\alpha = \oplus_{i=0}^\infty \mathcal{D}_i^\alpha \tag{93}$$

of first-order polydifferential operators putting inductively

$$(A \cdot B)(a) = (-1)^{\langle a+\alpha,B+j\alpha \rangle + j} A(a) \cdot B + A \cdot B(a) \tag{94}$$

for $a \in \mathcal{A}$, $A \in \mathcal{D}_i^\alpha$, $B \in \mathcal{D}_j^\alpha$. Similarly as in [Kr2], one checks that $A \cdot B$ is in $\mathcal{D}_{i+j}^\alpha$ and one proves the following.

Theorem 9. *The multiplication (94) turns the space $\mathcal{D}^\alpha(\mathcal{A})$ of first-order polydifferential operators on $\mathcal{A}$ into an $(n + 1)$-graded associative commutative unital algebra with homogeneous elements A from $\mathcal{D}_i^\alpha$ being of degree $(|A| + i\alpha, i)$. The graded subspace $\mathcal{DE}^\alpha(\mathcal{A})$ is a graded subalgebra of $\mathcal{D}^\alpha(\mathcal{A})$.*

Remark. The graded commutativity in the above theorem can be explicitly written in the form

$$A \cdot B = (-1)^{\langle A+i\alpha,B+j\alpha \rangle + ij} B \cdot A. \tag{95}$$

Here and later on we denote by A also the degree $|A|$ if no confusion arises. Note also that for $A, B \in \mathcal{D}_1^\alpha$ we get

$$(A \cdot B)(a,b) = (-1)^{\langle a+B,A+\alpha \rangle} B(a)A(b) - (-1)^{\langle a+\alpha,B+\alpha \rangle} A(a)B(b). \tag{96}$$

In particular, for the ungraded case,

$$A \cdot B(a,b) = B(a)A(b) - A(a)B(b) = (B \wedge A)(a,b). \tag{97}$$

Thus the defined product is the reversed wedge product.

The graded Schouten–Jacobi bracket on $\mathcal{D}^\alpha$ we define formally as in [Kr2] putting $[\![a, b]\!] = 0$ for $a, b \in \mathcal{A}$,

$$[\![D, a]\!] = D(a) \qquad [\![a, D]\!] = (-1)^{\langle a-\alpha, D+(i-1)\alpha\rangle + i} D(a) \tag{98}$$

for $a \in \mathcal{A}$, $D \in \mathcal{D}_i^\alpha$, and

$$[\![A, B]\!](a) = (-1)^{\langle a-\alpha, B+(j-1)\alpha\rangle + j-1} [\![A(a), B]\!] + [\![A, B(a)]\!] \tag{99}$$

for $a \in \mathcal{A}$, $A \in \mathcal{D}_i^\alpha$, $B \in \mathcal{D}_j^\alpha$. We get a graded Lie algebra structure on $\mathcal{D}^\alpha$ but, since now

$$[\![D, ab]\!] = D(ab) = D(a) \cdot b + (-1)^{\langle a, D+(i-1)\alpha\rangle} a \cdot D(b) - D(\mathbf{1}) \cdot ab \tag{100}$$

$$= [\![D, a]\!]b + (-1)^{\langle a, D+(i-1)\alpha\rangle} a \cdot [\![D, b]\!] - [\![D, \mathbf{1}]\!] \cdot ab \tag{101}$$

instead of the Leibniz rule we get its generalization.

Theorem 10. *There is a unique $(n + 1)$-graded Jacobi bracket $[\![\cdot, \cdot]\!]$ of degree $(-\alpha, -1)$ on the commutative algebra $\mathcal{D}^\alpha(\mathcal{A})$ of the previous theorem, satisfying (98). The associative subalgebra $\mathcal{DE}^\alpha(\mathcal{A})$ is a graded Jacobi subalgebra of $\mathcal{D}^\alpha(\mathcal{A})$.*

The above theorem tells that uniform elements D from $\mathcal{D}_i^\alpha$ have the degree $(|D|+(i-1)\alpha, i-1)$ with respect to the bracket. The generalized Leibniz rule reads

$$[\![A, B \cdot C]\!] = [\![A, B]\!] \cdot C + (-1)^{\langle A+(i-1)\alpha, B+j\alpha\rangle + (i-1)j} B \cdot [\![A, C]\!] - [\![A, \mathbf{1}]\!] \cdot B \cdot C \tag{102}$$

and the properties of the graded bracket, written explicitly, are

$$[\![A, B]\!] = -(-1)^{\langle A+(i-1)\alpha, B+(j-1)\alpha\rangle + (i-1)(j-1)} [\![B, A]\!] \tag{103}$$

$$[\![[\![A, B]\!], C]\!] = [\![A, [\![B, C]\!]]\!] - (-1)^{\langle A+(i-1)\alpha, B+(j-1)\alpha\rangle + (i-1)(j-1)} [\![B, [\![A, C]\!]]\!]. \tag{104}$$

Remark. It is obvious by definitions that if α and α' have the same parity, i.e. $\alpha - \alpha'$ has even coordinates, then $\mathcal{D}_i^\alpha(\mathcal{A})$ coincides with $\mathcal{D}_i^{\alpha'}(\mathcal{A})$ and the graded Jacobi algebras $\mathcal{D}^\alpha(\mathcal{A})$ and $\mathcal{D}^{\alpha'}(A)$ are isomorphic.

An element $S \in \mathcal{D}_2^\alpha(\mathcal{A})$ in the graded Jacobi algebra $(\mathcal{D}^\alpha(\mathcal{A}), \cdot, [\![\cdot, \cdot]\!])$ we call a *supercanonical structure* in $\mathcal{A}$ if

(i) $\langle |S| + \alpha, |S| + \alpha\rangle$ is an even number, and
(ii) $[\![S, S]\!] = 0$.

Similarly as in [Kr2], one proves that any supercanonical structure S in $\mathcal{A}$ determines a graded Jacobi bracket $\{\cdot, \cdot\}_S$ in the graded algebra $\mathcal{A}$ by

$$\{a, b\}_S = (-1)^{\langle a+\alpha, S+\alpha\rangle} S(a, b). \tag{105}$$

We have changed slightly the original definition by a sign in order to get the proper graded Lie algebra bracket. This time the bracket $\{\cdot, \cdot\}_S$ is a Jacobi and not a Poisson bracket in view of (102). Note that if the degree of S is just α, then (i) is satisfied automatically and $\{a, b\}_S = S(a, b)$. Thus we have the following.

Theorem 11. *For any supercanonical structure $S \in \mathcal{D}^\alpha(\mathcal{A})$ (resp. $S \in \mathcal{DE}^\alpha(\mathcal{A})$) the formula (105) defines a graded Jacobi (resp. graded Poisson) bracket on $\mathcal{A}$ of degree $|S|$. Conversely, every graded Jacobi (resp. Poisson) bracket $\{\cdot, \cdot\}$ of degree α on $\mathcal{A}$ determines by $S(a, b) = \{a, b\}$ a supercanonical structure S of $\mathcal{D}^\alpha(\mathcal{A})$ (resp. $\mathcal{DE}^\alpha(\mathcal{A})$).*

We will call this supercanonical structure $S \in \mathcal{D}^\alpha(\mathcal{A})$ *associated* with the graded Jacobi bracket of degree α. Having a graded Lie algebra structure on $\mathcal{D}^\alpha(\mathcal{A})$ and a supercanonical structure $S \in \mathcal{DE}^\alpha(\mathcal{A})$ (resp. $S \in \mathcal{D}^\alpha(\mathcal{A})$) we have, in a standard way, a cohomology operator $\partial_S = [\![S, \cdot]\!]$ which maps $\mathcal{DE}_i^\alpha(\mathcal{A})$ (resp. $\mathcal{D}_i^\alpha(\mathcal{A})$) into $\mathcal{DE}_{i+1}^\alpha(\mathcal{A})$ (resp. $\mathcal{D}_{i+1}^\alpha(\mathcal{A})$). This operator has the form of a 'Hamiltonian vector field'. The corresponding cohomology we will denote by $H_P^*(\mathcal{A}, S)$ (resp. $H_J^*(\mathcal{A}, S)$). For any Poisson (resp. Jacobi) bracket of degree α on $\mathcal{A}$ we have then the cohomology operator ∂_S and the corresponding cohomology $H_P^*(\mathcal{A}, S)$ (resp. $H_J^*(\mathcal{A}, J)$), for the associated supercanonical structure $S \in \mathcal{D}^\alpha(\mathcal{A})$, which we will call the *Poisson* (resp. *Jacobi*) *cohomology* of the graded Poisson (resp. Jacobi) algebra $(\mathcal{A}, \cdot, S)$.

We can deform canonically these cohomology operators in the spirit of Witten [Wi] as follows.

Lemma 3. *If $S \in \mathcal{D}^\alpha(\mathcal{A})$ is a supercanonical structure of degree α then*

(i) $[\![S, S(\mathbf{1})]\!] = 0$
(ii) $[\![S, S(\mathbf{1}) \cdot A]\!] = -S(\mathbf{1}) \cdot A$.

Proof. By definition, $0 = [\![S, S]\!](\mathbf{1}) = 2[\![S, S(\mathbf{1})]\!]$ and, in view of the generalized Leibniz rule,

$$[\![S, S(\mathbf{1}) \cdot A]\!] = [\![S, S(\mathbf{1})]\!] \cdot A - S(\mathbf{1}) \cdot [\![S, A]\!] - S(\mathbf{1}) \cdot S(\mathbf{1}) \cdot A = S(\mathbf{1}) \cdot [\![S, A]\!] \tag{106}$$

due to (i) and $S(\mathbf{1}) \cdot S(\mathbf{1}) = 0$. $\qquad\square$

It follows from the above lemma that

$$\partial_{\text{LJ}}^t(A) = [\![S, A]\!] + t S(\mathbf{1}) \cdot A \tag{107}$$

has square 0, i.e. it is a cohomology operator of degree $(\alpha, 1)$ on $\mathcal{D}^\alpha(\mathcal{A})$ for any parameter t. The operator ∂_{LJ}^0 is just ∂_S and ∂_{LJ}^1, denoted simply by ∂_{LJ}, we will call the *Lichnerowicz–Jacobi cohomology operator*. The associated cohomology will be denoted by $H_{\text{LJ}}^*(\mathcal{A}, S)$ and called *Lichnerowicz–Jacobi cohomology* of the algebra $\mathcal{A}$ associated with the Jacobi structure S.

Note that the general Jacobi brackets as above do not always split into biderivation and derivation, as in the classical case. There is a new interesting feature that there exist graded Jacobi brackets that are bidifferential operators of order 0.

Example 3. Consider the following bracket defined on the Grassmann algebra $\Omega(TM)$ of standard differential forms on M by $\{\alpha, \beta\}_\mu = \alpha \wedge \mu \wedge \beta$, where μ is a fixed 1-form. This bracket is clearly of order 0 as a bidifferential operator and of graded degree 1. Moreover,

$$\{\alpha, \beta\}_\mu = -(-1)^{(\alpha+1)(\beta+1)}\{\beta, \alpha\}_\mu \tag{108}$$

and any double bracket $\{\{\alpha, \beta\}_\mu, \gamma\}_\mu$ vanishes, so that the Jacobi identity is automatically satisfied. In other words, $\{\cdot, \cdot\}_\mu$ is a graded Jacobi bracket on the algebra of differential forms of order 0 and degree 1. In particular, $\mu = \{\mathbf{1}, \mathbf{1}\}_\mu$.

In fact, every graded Jacobi bracket $\{\cdot, \cdot\} = S$ of n-degree $\alpha \in \mathbb{Z}^n$ admits a more general decomposition

$$\{a, b\} = \Lambda(a, b) + (\Gamma \cdot I)(a, b) + (c \cdot I^2)(a, b) \tag{109}$$

$$= \Lambda(a, b) + a\Gamma(b) - (-1)^{\langle a+\alpha, \alpha\rangle}\Gamma(a)b + 2acb \tag{110}$$

for certain $\Lambda \in \mathcal{DE}_2^\alpha$, $\Gamma \in \mathcal{DE}_1^\alpha$, $c \in \mathcal{DE}_0^\alpha = \mathcal{A}$. Explicitly,

$$\Lambda = S - S(\mathbf{1}) \cdot I + \tfrac{1}{2}S(\mathbf{1}, \mathbf{1}) \cdot I^2 \qquad \Gamma = S(\mathbf{1}) - S(\mathbf{1}, \mathbf{1}) \cdot I \qquad c = \tfrac{1}{2}S(\mathbf{1}, \mathbf{1}). \tag{111}$$

Here, clearly, $I^2 = I \cdot I$. This can be generalized as follows.

Theorem 12. *Every $D \in \mathcal{D}_i^\alpha$ splits into*

$$D = D_0 + \frac{1}{1!} D_1 \cdot I + \cdots + \frac{1}{i!} D_i \cdot I^i \tag{112}$$

where

$$D_l = \sum_{k=0}^{i-l} (-1)^k D\underbrace{(\mathbf{1}, \ldots, \mathbf{1})}_{(k+l)\,\text{times}} \cdot \frac{I^k}{k!} \in \mathcal{DE}_l^\alpha(\mathcal{A}). \tag{113}$$

Proof. Note that, due to graded commutativity, $I \cdot I = 0$ and $D(\mathbf{1}, \mathbf{1}) = 0$ in the case when $\langle \alpha, \alpha \rangle$ is even. In the case when $\langle \alpha, \alpha \rangle$ is odd, one shows easily by induction that

$$(A \cdot I^n)(\mathbf{1}) = A(\mathbf{1}) \cdot I^n + nA \cdot I^{(n-1)}. \tag{114}$$

It is now just a matter of direct calculation to show that $D_l(\mathbf{1}) = 0$ and, using the identity

$$\sum_{k=0}^{s} \frac{1}{k!(s-k)!} = 0 \tag{115}$$

for $s > 0$, that the equation (112), with D_l defined by (113), is tautological. $\square$

Corollary 5. *The decomposition (112) determines an identification of $\mathcal{D}_i^\alpha(\mathcal{A})$ with*

 (i) $\mathcal{DE}_i^\alpha(\mathcal{A}) \oplus \mathcal{DE}_{i-1}^\alpha(\mathcal{A})$ *in the case when $\langle \alpha, \alpha \rangle$ is even;*
(ii) $\mathcal{DE}_i^\alpha(\mathcal{A}) \oplus \mathcal{DE}_{i-1}^\alpha(\mathcal{A}) \oplus \ldots \oplus \mathcal{DE}_0^\alpha(\mathcal{A})$ *in the case when $\langle \alpha, \alpha \rangle$ is odd.*

Proof. In the case when $\langle \alpha, \alpha \rangle$ is even, $I \cdot I = 0$, so that the splitting (112) reduces to two terms. In the other case, all I^k are non-zero, due to the formula $I^k(a) = kI^{(k-1)} \cdot a$ which can easily be proved by induction. $\square$

Using these decompositions one can describe the bracket in $\mathcal{D}^\alpha$ in terms of the bracket in $\mathcal{DE}^\alpha$. In the case when $\langle \alpha, \alpha, \rangle$ is even it is completely analogueous to the ungraded case (see the next example). For the other case it is sufficient to use the following lemma.

Lemma 4. *Suppose that $\langle \alpha, \alpha, \rangle$ is odd. Then,*

$$[\![A \cdot I^n, B \cdot I^m]\!] = (-1)^{\langle (n-1)\alpha, B \rangle} (m(i+1) - n(j+1)) A \cdot B \cdot I^{n+m-1}$$
$$+ (-1)^{\langle n\alpha, B \rangle} [\![A, B]\!] \cdot I^{n+m}. \tag{116}$$

Example 4. For the (ungraded) algebra $\mathcal{A} = C^\infty(M)$ the graded Poisson algebra $\mathcal{DE}^\alpha(\mathcal{A}) = \mathcal{DE}(\mathcal{A})$ is simply the Gerstenhaber algebra $A(TM)$ of multivector fields with the (reversed) wedge product and the Schouten bracket. The graded Jacobi algebra $\mathcal{D}^\alpha(\mathcal{A}) = \mathcal{D}(\mathcal{A})$ is in this case the Grassmann algebra $A(TM \oplus \mathbb{R}) = \mathrm{Sec}(\wedge(TM \oplus \mathbb{R}))$ with the bracket described in [GM]. Since here $I \cdot I = 0$, we have the identification

$$\mathcal{DE}_i(\mathcal{A}) = \mathcal{DE}_i(\mathcal{A}) \oplus \mathcal{DE}_{i-1}(\mathcal{A}) = A^i(TM) \oplus A^{i-1}(TM). \tag{117}$$

The Schouten–Jacobi bracket reads (cf [GM], formula (27)):

$$[\![A_1 + I \wedge A_2, B_1 + I \wedge B_2]\!] = [A_1, B_1] + (-1)^a I \wedge [A_1, B_2] + I \wedge [A_2, B_1] \tag{118}$$

$$+ a A_1 \wedge B_2 - (-1)^a b A_2 \wedge B_1 + (a-b) I \wedge A_2 \wedge B_2 \tag{119}$$

for $A \in \mathcal{D}_{a+1}^\alpha(\mathcal{A})$, $B \in \mathcal{D}_{b+1}^\alpha(\mathcal{A})$. Here $[\cdot, \cdot]$ is the Schouten–Nijenhuis bracket and we use the standard wedge product. For a Jacobi bracket $S = \Lambda + I \wedge \Gamma$ this gives the cohomology operator

$$\partial_S((B_1, B_2)) = ([\Lambda, B_1] + \Lambda \wedge B_2 + b\Gamma \wedge B_1, -[\Lambda, B_2] + [\Gamma, B_1] + (1 - b)\Gamma \wedge B_2). \quad (120)$$

This is exactly (up to differences in conventions of signs for the Schouten bracket) the cohomology operator of 1-differentiable Chevalley–Eilenberg cohomology of the Jacobi bracket on $C^\infty(M)$, introduced and considered in [Li]. The Lichnerowicz–Jacobi cohomology operator is in this case

$$\partial_{LJ}((B_1, B_2)) = ([\Lambda, B_1] + \Lambda \wedge B_2 + (b + 1)\Gamma \wedge B_1, -[\Lambda, B_2] + [\Gamma, B_1] - b\Gamma \wedge B_2). \quad (121)$$

This cohomology has been extensively studied in [LMP, LLMP] and we refer to [LLMP] for more particular examples and explicit calculations of particular cohomology.

Example 5. Let $\mathcal{A} = A(E)$ be the Grassmann algebra of a vector bundle E. The graded Jacobi bracket (7) on $A(E)$ corresponds to a supercanonical structure $S \in \mathcal{D}^{-1}(A(E))$. The decomposition (109) in this case reads

$$S = S_0 + i_\Phi \cdot \mathrm{Deg} - i_\Phi \cdot I \qquad (122)$$

where S_0 is the supercanonical structure corresponding to the Schouten–Nijenhuis bracket.

Remark. Suppose we start not from a graded algebra but from just an n-graded vector space V. We can define inductively spaces $\mathbf{A}_i^\alpha(V)$ of multilinear maps in V similarly to $\mathcal{D}_i^\alpha(\mathcal{A})$ just relaxing the assumption on the generalized Leibniz rule. Then we can define a graded Lie bracket $[\![\cdot, \cdot]\!]^{NR}$ on $\mathbf{A}^\alpha(V) = \oplus_{i=0}^\infty \mathbf{A}_i^\alpha(V)$ completely along the same lines as the Jacobi bracket on $\mathcal{D}^\alpha(\mathcal{A})$. It is just a graded Lie bracket satisfying (103) and (104) but the Leibniz rule has no meaning. We will call this bracket the *Nijenhuis–Richardson bracket* of multilinear maps in V. For a graded algebra $\mathcal{A}$ the space $\mathcal{D}^\alpha(\mathcal{A})$ is just a Lie subalgebra in $\mathbf{A}^\alpha(\mathcal{A})$. It can be shown that this is exactly the Nijenhuis–Richardson bracket defined in [LMS] for the graded vector space V with the grading shifted by α.

6. Conclusions

The notion of a Jacobi algebra turns out to unify various concepts of algebra and differential geometry. A particularly interesting case is a linear Jacobi bracket on the exterior algebra of a vector bundle, i.e. a Jacobi algebroid. Using a gauging method we have obtained general cohomology and homology theories which include a whole spectrum of (co)homology associated with classical Poisson and Jacobi structures. Along these lines we have extended also the notion of a Courant algebroid. Our concept agrees with the known generalizations for the case of the tangent bundle.

The construction of a canonical Jacobi algebra $\mathcal{D}^\alpha(\mathcal{A})$ associated with a given graded commutative algebra $\mathcal{A}$ generalizes Schouten brackets and allows us to view the corresponding cohomology operators as 'Hamiltonian vector fields' of a supercanonical structure.

One can develop this algebraic theory further by defining an appropriate dual object to $\mathcal{D}^\alpha(\mathcal{A})$ (forms) and the corresponding Cartan–Jacobi differential calculus. We postpone these problems to a separate paper.

Acknowledgments

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